Lecture 11

- Recap
- Group of rotations
- Failure of the G. inv. of wave equation
- Mink. metric, upper / lower indices
- Addition of velocities, interval

Review

	Translations	Rotations	Galilean	Lorentz
Mechanics	✓	✓	✓	to be discussed
CED	✓	✓	✗	✓

Rotations:

$$\hat{R} = \underbrace{\hat{R}_{12}\,\hat{R}_{13}\,\hat{R}_{23}}^{SO(3)}$$

$$\hat{R}_{12} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

rotation in $x_1 \cup x_2$ plane

$$(\vec{x}, t) \rightarrow x^\mu : \begin{pmatrix} ct \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \text{4-vector}$$

$$ct \equiv x^0$$

Minkowski metric, upper/lower indices

If we worked with Euclidean time, four-vectors $x^\mu_E = (x^0_E, \vec{x})$ would work in exactly the same way as three-vectors:

$$\vec{x}\cdot\vec{x} = x_1^2 + x_2^2 + x_3^2$$

In reality we want to work with the real time, so every time the zeroth component appears, we have to change sign (since it appears squared due to Lorentz invariance)

$$x^\mu_E \cdot x^\mu_E = {x^0_E}^2 + {x^1}^2 + {x^2}^2 + {x^3}^2$$

hence, for Lorentz-invariant product of Lorentzian four-vectors we need

$$x^\mu \cdot x^\mu = -{x^0}^2 + {x^1}^2 + {x^2}^2 + {x^3}^2 \equiv x_\mu x^\mu,$$

where:

$$\boxed{x_0 \equiv -x^0, \quad x_1 = x^1, \quad x_2 = x^2, \quad x_3 = x^3}$$

To raise and lower indices we define
the Minkowski metric:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$x_\mu = \eta_{\mu\nu} x^\nu = (-x^0, x^1, x^2, x^3)$$
$$\parallel$$
$$\hat{\eta} \cdot x^\nu$$

Note: sometimes the opposite conventions are used:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

Expressions with contracted upper/lower indices are always Lorentz invariant!

- we treat $\frac{\partial}{\partial x^\mu}$ as an object with lower index: ∂_μ

$$\Lambda_\mu{}^\nu \Lambda^\rho{}_\nu = \mathbb{1}_\mu{}^\rho \qquad \Lambda_\mu{}^\nu \Lambda_{\rho\nu} = \eta_{\mu\rho}$$

check:
$$\begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} \begin{pmatrix} \cosh x & -\sinh x \\ -\sinh x & \cosh x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Lambda_\nu{}^\mu \Lambda^\nu{}_\rho = \mathbb{1}^\mu{}_\rho$$

$$x^\mu \to x'^\nu = \Lambda^\nu{}_\rho x^\rho \Rightarrow \Lambda_\nu{}^\mu x'^\nu = x^\mu$$

$$\phi'(x') = \phi(x(x'))$$, lets see how derivative transform.

$$\partial'_\nu \phi' = \frac{\partial}{\partial x'^\nu} \phi(x(x')) = \frac{\partial}{\partial x^\mu} \phi(x) \cdot \frac{\partial x^\mu}{\partial x'^\nu} =$$

$$= \Lambda_\nu{}^\mu \frac{\partial}{\partial x^\mu} \phi(x) = \Lambda_\nu{}^\mu \partial_\mu \phi$$

$$\Box = -\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\eta^{\mu\nu} \rightarrow \text{Lorentz inv.}$$

Lorentz transform: $x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu$, here $\Lambda^\mu{}_\nu$ is the matrix $\hat{\Lambda}$. We can also raise and lower its indices by $\eta_{\mu\nu}$.

Vector with lower indices transforms as

$$x_\mu' = \Lambda_\mu{}^\nu x_\nu$$

Usual multiplication of matrices corresponds to contracting tensors with one lower, one upper index. For example, composition of two Lorentz transformations:

$$\Lambda^{(1)\mu}{}_\nu \Lambda^{(2)\nu}{}_\rho = \Lambda^{(3)\mu}{}_\rho$$

- More general 4-tensors with arbitrary number of indices transform following the same rules as 4-vectors:

$$A_{\mu\nu} \to \Lambda_\mu{}^\rho \Lambda_\nu{}^\gamma A_{\rho\gamma}$$

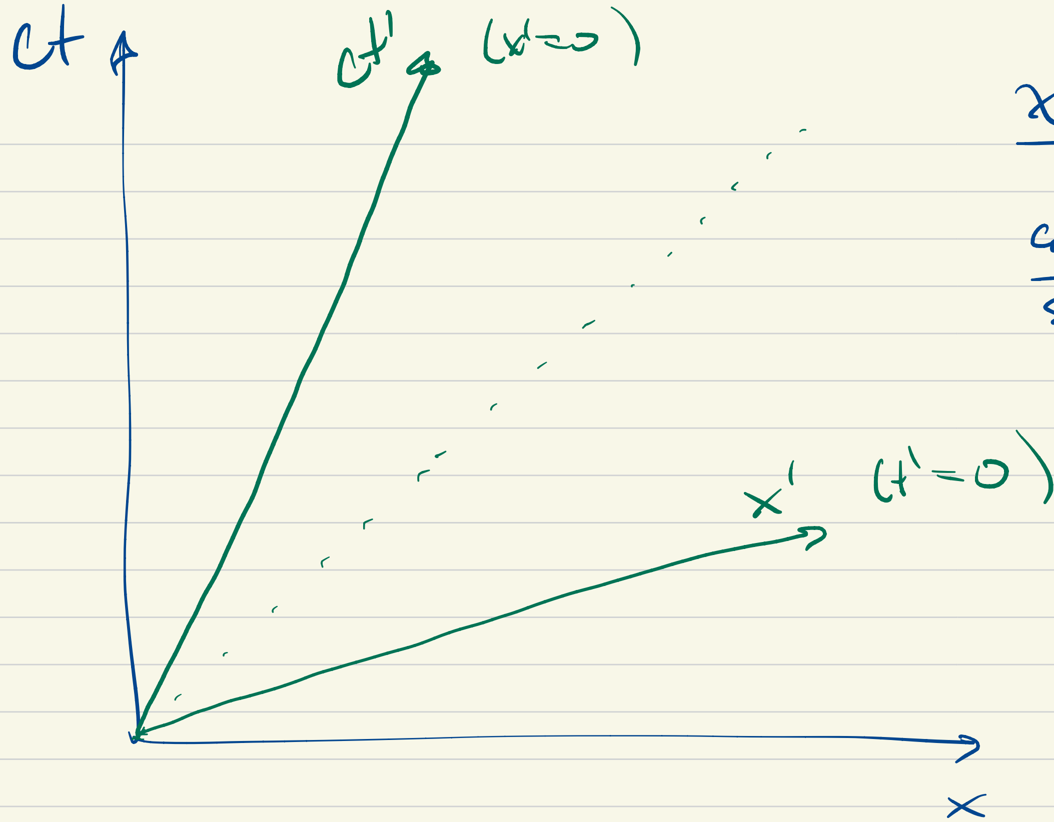
under Lorentz

Addition of velocities

$$\begin{cases} ct' = ct\cosh\chi + x_1\sinh\chi \\ x_1' = x_1\cosh\chi + ct\sinh\chi \\ x_2' = x_2 \\ x_3' = x_3 \end{cases}$$

$$x' = 0 \Rightarrow ct = -x\,\frac{\cosh\chi}{\sinh\chi}$$

$$t' = 0 \Rightarrow ct = -x\,\frac{\sinh\chi}{\cosh\chi}$$



$\underline{\chi < 0}$

$$\frac{\cosh\chi}{\sinh\chi} \geq 1$$

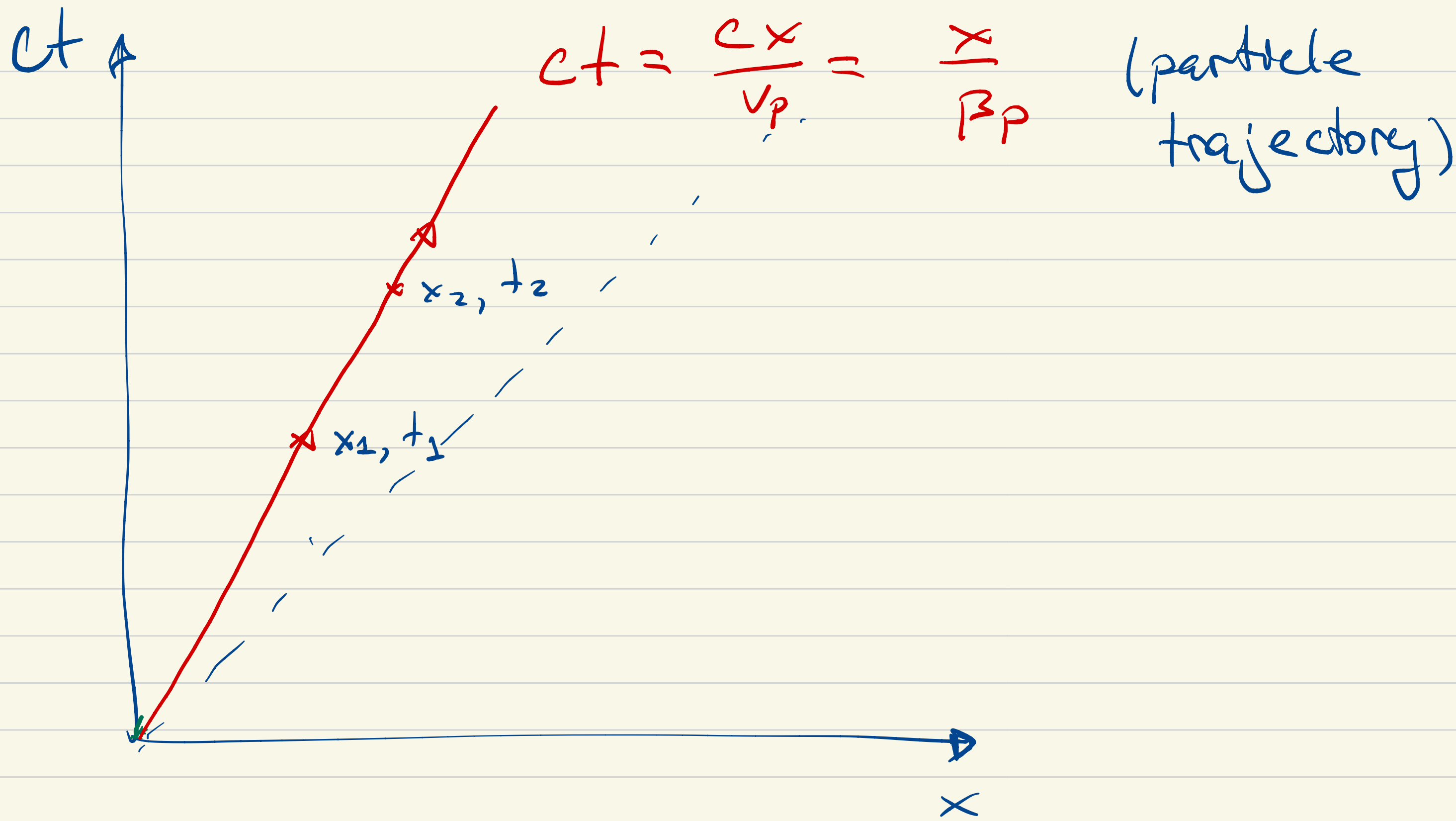
$$x' = 0 \implies x = -ct\,\frac{\sinh\chi}{\cosh\chi} \equiv vt$$

$$\frac{v}{c} \equiv -\frac{\sinh\chi}{\cosh\chi} \equiv \beta \quad (\text{rapidity})$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad [\text{boost}] \quad \gamma \geq 1$$

$$ct' = \gamma(ct - \beta x)$$

$$x' = \gamma(x - \beta ct)$$



what is the velocity of the particle in the new reference frame?

$$x_2, t_2 \rightarrow x_2', t_2'$$

$$x_1, t_2 \rightarrow x_1', t_1'$$

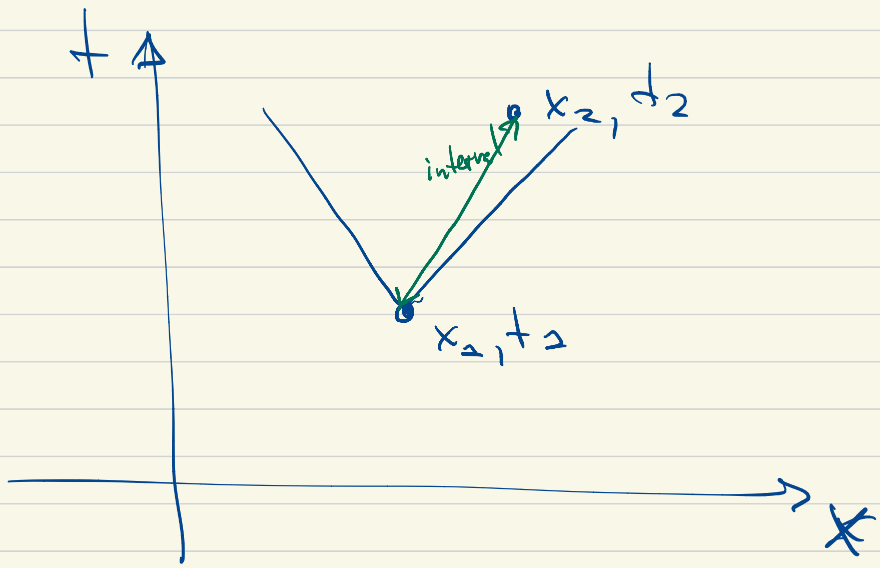
$$\frac{v_p'}{c} = \frac{x_2' - x_1'}{c(t_2' - t_1')} = \frac{\gamma\left((x_2 - x_1) - \beta c(t_2 - t_1)\right)}{\gamma\left((ct_2 - ct_1) - \beta(x_2 - x_1)\right)}$$

$$= \frac{v_p - \beta c}{c - \beta v_p} \Rightarrow v_p' = \frac{v_p - v}{1 - \frac{v v_p}{c^2}}$$

$$v_p = c \Rightarrow v_p' = c \, !$$

Interval

$$s^2 = -c^2 \Delta t^2 + (\Delta \vec{x})^2 = \Delta x^\mu \Delta x_\mu$$



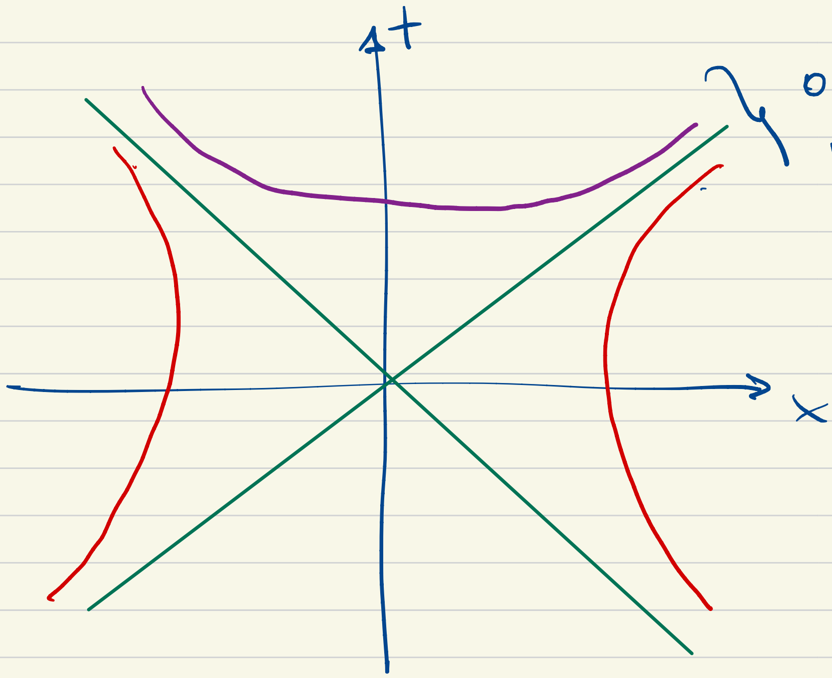
- interval corresponds to the "internal" or proper time of the object

- Time inside the objects moving with some speed seems to be slowed down when observed from the outside

$s^2 < 0 \Rightarrow$ time-like interval (signal can reach)

$s^2 > 0 \Rightarrow$ space-like interval (signal cannot reach)

$s^2 = 0 \Rightarrow$ null interval ~ lightcone



orbits of the Lorentz group

Transformation of Φ and $\vec{A}$, Lorentz inv. and Maxwell equations

First, let us study Galilean transformations of the charge density and current density: $t(t',\vec{x}')$, $\vec{x}(t',x')$

$$\rho(t,\vec{x}) \to \rho'(t',\vec{x}') = \rho(\overbrace{t,\vec{x}})$$

$$\vec{J}(t,x) \to \vec{J}(t,x) + \vec{v}\rho(t,x)$$

this looks like a 4-vector transformation

$$t \to t' = t$$

$$\vec{x} \to \vec{x}' = \vec{x} + \vec{v}\cdot t$$

$(c\rho, \vec{J}) \equiv J^\mu \to$ 4-vector-valued function: $J^\mu(x^\mu)$

It is natural to expect that it transforms as a four-vector under full relativistic transformations:

$$J^\mu(x^\mu) \longrightarrow \Lambda^\mu{}_\rho J^\rho(x^\rho(x))$$

Charge conservation is Lorentz inv.:

$$\partial_\mu J^\mu = \frac{c}{c}\partial_t \rho + \vec{\nabla}\cdot\vec{J} = 0$$

$$\begin{aligned} \Box \varphi &= \frac{1}{\varepsilon_0}\rho \\ \Box \vec{A} &= \mu_0 \vec{J} \end{aligned} \quad \Rightarrow \quad \Box \begin{pmatrix} \varphi \\ c\vec{A} \end{pmatrix} = \frac{1}{c\varepsilon_0}\begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}$$

To maintain Lorentz invariance $(\varphi, c\vec{A})$ should also transform as a 4-vector: $A^\mu(x^\mu)$

different people!

Lorenz / gauge condition is also Lorentz invariant:

$$\partial_\mu A^\mu = 0 \Rightarrow \frac{\partial_t \varphi}{c} + c \nabla_i A_i = 0$$

Gauge symmetry is Lorentz covariant:

$$A^\mu = (\varphi, c\vec{A})$$

$$A_\mu \rightarrow A_\mu + c\,\partial_\mu \alpha$$

$$\varphi^0 \rightarrow \varphi^0 - \frac{\partial}{\partial t}\alpha \;, \quad \vec{A} \rightarrow \vec{A} + \partial_i \alpha$$

So we demonstrated full Lorentz invariance of Maxwell equations!

Next lecture we will derive the covariant form of Maxwell equations also in terms of $\vec{E}$ and $\vec{B}$ (it will be very nice!)